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LETTER TO THE EDITOR

Non-linear differential-difference equations with N-dependent coefficients I†

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Abstract. Employing the AKNS technique, we generalise the discrete non-linear evolution equations obtained from the four-potential discrete Zakharov and Shabat spectral problem to include equations with n-dependent coefficients.

The most general discrete Zakharov and Shabat (ZS) spectral problem contains four potentials, as shown by Ablowitz (1978); there is a subcase, extensively treated in literature (Chiu and Ladik 1977, Levi and Ragnisco 1978), which contains only two potentials, but for which we are obliged to impose restrictions on the spectral data (Ablowitz 1978). For this subcase, following the works of Newell (1978) and Calogero and Degasperis (1978), we have been able to derive a class of non-linear differential-difference equations (NDDE) with n-dependent coefficients (Levi and Ragnisco 1978), by introducing a new Wronskian relation. It seems worthwhile to derive the class of NDDE with n-dependent coefficients for the more general ZS spectral problem. Thus we turned to the Ablowitz, Kaup, Newell and Segur (Ablowitz *et al* 1974) (AKNS) method, extended to the discrete case by Ablowitz and Ladik (1975), which, through an algebraic procedure, allows one to obtain (at least) the simplest equations of the class.

The four-potential discrete ZS spectral problem reads

$$\begin{aligned} v_{1,n+1}(z,t) &= zv_{1,n}(z,t)+Q_n(t)v_{2,n}(z,t)+S_n(t)v_{2,n+1}(z,t) \\ v_{2,n+1}(z,t) &= (1/z)v_{2,n}(z,t)+R_n(t)v_{1,n}(z,t)+T_n(t)v_{1,n+1}(z,t) \end{aligned} \tag{1}$$

where Q_n, R_n, S_n, T_n are the time dependent potentials and z is the 'eigenvalue'.

Generalising the AKNS procedure, we assume the following time dipendence for the eigenfunction

$$\boldsymbol{v}_n(z,t)=\begin{pmatrix} v_{1,n} \\ v_{2,n} \end{pmatrix}$$

$$(\partial/\partial t)\boldsymbol{v}_n(z,t)=\mathbf{L}_n(z,t)\boldsymbol{v}_n(z,t)+\mathbf{M}_n(z,t)(\partial/\partial z)\boldsymbol{v}_n(z,t) \tag{2}$$

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where $\mathbf{L}_n$ and $\mathbf{M}_n$ are 2×2 matrices:

$$\mathbf{L}_n=\begin{pmatrix}A_n & B_n\\ C_n & D_n\end{pmatrix},\qquad \mathbf{M}_n=\begin{pmatrix}E_n & F_n\\ G_n & H_n\end{pmatrix}.$$

Imposing the integrability condition, i.e. that the time derivative of (1) be equal to (2) evaluated at $n+1$, we first obtain,

$$F_n=G_n=0;\qquad E_n=H_n=\mu(z,t)\ (\text{not depending on } n!) \tag{3}$$

and consequently that the following equations must be satisfied:

$$\begin{aligned}
&(S_nR_n)_t-(z+S_nR_n)(\Delta A_n-(\ln\Lambda_n)_t]+(Q_n+S_n/z)C_n-(R_n+zT_n)B_{n+1}-\mu=0;\\
&(Q_n+S_n/z)_t+(z+S_nR_n)B_n+(Q_n+S_n/z)[D_n-A_{n+1}-(\ln\Lambda)_t]\\
&\qquad+(z^{-1}+Q_nT_n)B_{n+1}+(S_n/z^2)\mu=0;\\
&(R_n+zT_n)_t+(R_n+zT_n)[A_n-D_{n+1}-(\ln\Lambda_n)_t]+C_n(z^{-1}+Q_nT_n)\\
&\qquad-C_{n+1}(z+S_nR_n)-T_n\mu=0;\\
&(Q_nT_n)_t+(R_n+zT_n)B_n-(z^{-1}+Q_nT_n)[\Delta D_n+(\ln\Lambda_n)_n)_t]\\
&\qquad-(Q_n+S_n/z)C_{n+1}+\mu/z^2=0,
\end{aligned}\tag{4}$$

where $\Lambda_n=1-S_nT_n$, $(\ldots)_t=(\partial/\partial t)(\ldots)$ and $\Delta(\ldots)_n=(\ldots)_{n+1}-(\ldots)_n$.

To deduce the simplest non-trivial differential-difference equations, we expand $\mathbf{L}_n$ and μ in powers of z and $1/z$ as follows:

$$\begin{aligned}
A_n(z,t)&=A_n^0(t)+A_n^1(t)z+A_n^{-1}(t)z^{-1};\\
B_n(z,t)&=B_n^0(t)+B_n^1(t)z+B_n^{-1}(t)z^{-1};\\
C_n(z,t)&=C_n^0(t)+C_n^1(t)z+C_n^{-1}(t)z^{-1};\\
D_n(z,t)&=D_n^0(t)+D_n^1(t)z+D_n^{-1}(t)z^{-1};\\
\mu(z,t)&=\mu^0(t)+\mu^1(t)z+\mu^2(t)z^2.
\end{aligned}$$

With this assumed form for $\mathbf{L}_n$ and μ, equations (4) yield for the different powers of z ($z^2, z, z^0, z^{-1}, z^{-2}$) a system of equations which must be independently satisfied, and can be solved by a straightforward although tedious algebraic manipulation. We obtain: (With no restriction we have set $A^{-1}=D^1=0$; furthermore, we remark that the coefficients A^0, A^1, etc appearing in equations (5) may depend on time but *not on n*.)

$$\begin{aligned}
A_n(z,t)&=A^0+A^1z-n\mu z^{-1}-\sum_{j=n}^{\infty}[(D^{-1}+(2j+3)\mu^0)R_{j+1}S_j-(A^1-(2j+2)\mu^2)R_jS_j\\
&\qquad-(D^{-1}+(2j+1)\mu^0)Q_jT_j+(A^1-2j\mu^2)Q_jT_{j-1}];\\
B_n(z,t)&=(A^1-(2n+1)\mu^2)Q_n+(D^{-1}+2n\mu^0)S_{n-1}/z;\\
C_n(z,t)&=(D^1+(2n+1)\mu^0)R_n+(A^1-2n\mu^0)T_{n-1}z;\\
D_n(z,t)&=D^0+D^{-1}/z+n\mu z^{-1}-\sum_{j=n}^{\infty}[(D^{-1}+2j\mu^0)R_jS_{j-1}-(A^1-(2j+1)\mu^2)R_jS_j\\
&\qquad+(A^1-(2j+3)\mu^2)Q_{j+1}T_j-(D^{-1}+(2j+2)\mu^0)Q_jT_j],
\end{aligned}\tag{5}$$

together with the evolution equations

$$
\begin{aligned}
Q_{n,t} &= -[D^0 - A^0 + 2(n+1)\mu^1 + \mu^0 Q_n T_n + \mu^2 R_n S_n + I_{n+1}]Q_n \\
&\quad + [(A^1 - 2(n+1)\mu^2)S_n - (D^{-1} + 2n\mu^0)S_{n-1}](1 - Q_n R_n); \\
R_{n,t} &= [D^0 - A^0 + 2(n+1)\mu^1 + \mu^0 Q_n T_n + \mu^2 R_n S_n + I_{n+1}]R_n \\
&\quad + [(D^{-1} + 2(n+1)\mu^0)T_n - (A^1 - 2n\mu^2)T_{n-1}](1 - Q_n R_n); \\
S_{n,t} &= -[D^0 - A^0 + 2(n+1)\mu^1 + I_{n+1}]S_n \\
&\quad + [(A^{-1} - (2n+3)\mu^2)Q_{n+1} - (D^{-1} + (2n+1)\mu^0)Q_n](1 - S_n T_n); \\
T_{n,t} &= [D^0 - A^0 + 2(n+1)\mu^1 + I_{n+1}]T_n \\
&\quad + [(D^{-1} + (2n+3)\mu^0)R_{n+1} - (A^1 - (2n+1)\mu^2)R_n](1 - S_n T_n),
\end{aligned}
\tag{6}
$$

where

$$
I_k = \sum_{j=k}^{\infty} [\mu^2(R_j S_j + Q_j T_{j-1}) + \mu^0(R_j S_{j-1} + Q_j T_j)].
$$

Special subcases are easily obtained:

(a) $R_n = \epsilon Q_n$; $\qquad T_n = \epsilon S_n (\epsilon = \pm 1)$.

This case is consistent if $D^0 = A^0, A^1 = D^{-1}, \mu^0 = -\mu^2, \mu^1 = 0$. The evolution equations read:

$$
\begin{aligned}
Q_{n,t} &= (1 - \epsilon Q_n^2)[(A^1 + 2(n+1)\mu^0)S_n - (A^1 + 2n\mu^0)S_{n-1}]; \\
S_{n,t} &= (1 - \epsilon S_n^2)[(A^1 + (2n+3)\mu^0)Q_{n+1} - (A^1 + (2n+1)\mu^0)Q_n].
\end{aligned}
\tag{7}
$$

Setting $W_{2m} = S_m$, $W_{2m-1} = Q_m$ we can obtain a generalised discrete modified Korteweg–De Vries equation (Bruschi *et al* 1978)

$$
W_{n,t} = (1 - \epsilon W_n^2)[(A^1 + \mu^0(n+3))W_{n+1} - (A^1 + (n+1)\mu^0)W_{n-1}] \tag{8}
$$

from which a generalised Volterra system can be derived:

$$
\begin{aligned}
N_{n,t} &= \alpha N_n[(2n - 3 + \gamma)N_{n-1} - (2n + 3 + \gamma)N_{n+1}] \\
&\quad - 8\epsilon\alpha N_n - 2\alpha N_n^2 \qquad (\alpha = \epsilon\mu^0/2, \gamma = 3 + 2A^1/\mu^0).
\end{aligned}
\tag{9}
$$

(b) $R_n = \epsilon Q_n^*$, $\qquad T_n = \epsilon S_n^* \qquad (\epsilon = \pm 1)$.

This case is consistent if $(A^1)^* = D^{-1}, D^0 - A^0 = -(D^0 - A^0)^*, \mu^1 = -\mu_1^*, \mu_2 = -\mu_0^*$. The corresponding non-linear Schrödinger-like equations are:

$$
\begin{aligned}
Q_{n,t} &= -\mathrm{i}\,\mathrm{Im}\,[D^0 - A^0 + 2(n+1)\mu^1 + 2\epsilon\mu^0(J_{n+1} + Q_n S_n^*)]Q_n \\
&\quad + (1 - \epsilon|Q_n|^2)[(A^1 + 2(n+1)\mu_0^*)S_n - (A^{1*} + 2n\mu^0)S_{n-1}]; \\
S_{n,t} &= -\mathrm{i}\,\mathrm{Im}\,[D^0 - A^0 + 2(n+1)\mu^1 + 2\epsilon\mu^0 J_{n+1}]S_n \\
&\quad + (1 - \epsilon|S_n|^2)[(A^1 + (2n+3)\mu_1^{0*})Q_{n+1} - (A^{1*} + (2n+1)\mu^0)Q_n]
\end{aligned}
\tag{10}
$$

where

$$
J_k = \sum_{j=k}^{\infty} (Q_j S_j^* + Q_j^* S_{j-1}).
$$

(c) $R_n = 0$, $T_n = 1$:

This case is consistent if $\mu^1 = 0$, $A^1 = \mu^0$, $D^{-1} = -\mu^0$, $D^0 = A^0$, $\mu^2 = -\mu^0$ (μ^0 not time dependent). Setting $Q_n = 2\beta_n$, $S_n = 1 - 4\alpha_n$, equations (6) can be cast in the n-dependent Toda lattice–form (Toda 1976):

$$\begin{aligned}\alpha_{n,t} &= 2A^1\alpha_n(\beta_n - \beta_{n+1}) + 2\mu^0\alpha_n[(n-1)\beta_n - (n+1)\beta_{n+1}],\\ \beta_{n,t} &= 2A^1(\alpha_{n-1} - \alpha_n) + 2\mu^0[1 - \beta_n^2 + 2(n-1)\alpha_{n-1} - 2(n+1)\alpha_n].\end{aligned} \tag{11}$$

Setting

$$\alpha_n = \exp\{2A^1(W_n - W_{n+1})\}\exp\{2\mu^0[(n-1)W_n - (n+1)W_{n+1}]\},$$

it follows that $\beta_n = W_{n,t}$; consequently we obtain the second-order differential difference equation for the 'n-dependent Toda lattice' with a velocity-dependent friction term:

$$\begin{aligned}W_{n,tt} = 2A^1(&\exp\{2[(n-2)\mu^0 + A^1]W_{n-1} - 2[n\mu^0 + A^1]W_n\}\\ &-\exp\{2[(n-1)\mu^0 + A^1]W_n - 2[(n+1)\mu^0 + A^1]W_{n+1}\})\\ &+2\mu^0(1 - W_{n,t}^2 + 2(n-1)\exp\{2[(n-2)\mu^0 + A^1]W_{n-1} - 2[n\mu_0 + A^1]W_n\}\\ &-2(n+1)\exp\{2[(n-1)\mu^0 + A^1]W_n - 2[(n+1)\mu^0 + A^1]W_{n+1}\}).\end{aligned} \tag{12}$$

Equations (6) can be solved by the Inverse Spectral Transform (IST) (Ablowitz and Ladik 1975), provided the potentials vanish sufficiently quickly as n goes to infinity. In this case, we can define the spectral data associated with equations (1):

$$\begin{aligned}S: \{&z_{(k)}^{(+)}(|z_{(k)}^{(+)}| > 1), C_{(k)}^{(+)}; z_{(k)}^{(-)}(|z_{(k)}^{(-)}| < 1), C_{(k)}^{(-)}(k = 1, \ldots, N);\\ &\times \beta^+(z)(|z| \geq 1), \beta^-(z)(|z| \leq 1)\}\end{aligned} \tag{13}$$

through the asymptotic behaviour, for the bound states

$$\lim_{n\to+\infty}[\phi_{(k)}^{\pm}(n,t) - C_{(k)}^{(\pm)}\chi^{\mp}(z_{(k)}^{(\pm)})^{\mp n}] = 0; \qquad \lim_{n\to-\infty}[\phi_{(k)}^{\pm}(n,t) - \bar{C}_{(k)}^{(\pm)}\chi^{\pm}(z_{(k)}^{(\pm)})^{\pm n}] = 0 \tag{14a}$$

and for the scattering states

$$\begin{aligned}\lim_{n\to+\infty}\left[\Phi(n,t) - \begin{pmatrix} z^n & \beta^-(z)z^n \\ \beta^+(z)z^{-n} & z^{-n}\end{pmatrix}\right] &= 0,\\ \lim_{n\to-\infty}\left[\Phi(n,t) - \begin{pmatrix} \alpha^+(z)z^n & 0 \\ 0 & \alpha^-(z)z^{-n}\end{pmatrix}\right] &= 0,\end{aligned} \tag{14b}$$

where $\phi_{(k)}^{(\pm)}$ are the normalised bound-state vectors, $\chi^{\pm}$ being the basic spinors

$$\chi^+ = \begin{pmatrix}1\\0\end{pmatrix}, \chi^- = \begin{pmatrix}0\\1\end{pmatrix},$$

and $C_{(k)}^{(\pm)}$, defined by equation (14a), are the residues of the reflection coefficients $\beta^{\pm}(z)$ at the poles $z = z_{(k)}^{(\pm)}$, for potentials vanishing faster than exponentially as $|n| \to \infty$;

$$C_{(k)}^{(\pm)} = \lim_{z\to z_{(k)}^{(\pm)}}[z - z_{(k)}^{(\pm)}]\beta^{(\pm)}(z). \tag{15}$$

Starting from the spectral data S (equation 13), one can recover in a unique way the potentials, by solving the Gelfand–Levitan–Marchenko equation:

$$\mathbf{K}(n, n') + \mathbf{M}(n+n') + \sum_{n''=n+1}^{\infty} \mathbf{K}(n, n'')\sigma_1 \mathbf{M}(n''+n') = 0 \tag{16a}$$

where $\mathbf{K}(n, n')$ is a 2×2 matrix,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{M}(n) = \begin{pmatrix} F^{(-)}(n) & 0 \\ 0 & F^{(+)}(n) \end{pmatrix}, \tag{16b}$$

$$F^{(\pm)}(n) = \pm(2\pi \mathrm{i})^{-1} \oint \mathrm{d}z\, \beta^{\pm}(z) z^{\mp n-1} + \sum_{k=1}^{N} C_{(k)}^{(\pm)} (z_{(k)}^{(\pm)})^{\mp n-1}, \tag{16c}$$

$$Q_n = -K_{11}(n_1 n + 1); \qquad R_n = -K_{22}(n_1 n + 1),$$

$$S_n = -(1 - R_n Q_n)^{-1}[K_{11}(n_1 n + 2) - K_{11}(n_1 n + 1) K_{21}(n_1 n + 1)], \tag{16d}$$

$$T_n = -(1 - R_n Q_n)^{-1}[K_{22}(n_1 n + 2) - K_{22}(n_1 n + 1) K_{12}(n_1 n + 1)].$$

We end this Letter by deducing the time evolution of the spectral data; considering equation (2) and equations (5) in the asymptotic region ($|n| \to \infty$), taking care of the asymptotic behaviour of the potentials, we obtain:

$$\beta_t^{\pm} = \pm\omega(z, t)\beta^{\pm} + \mu(z, t)\beta_z^{\pm}, \tag{17a}$$

$$\alpha_t^{\pm} = -I^{\pm}(t)\alpha^{\pm} + \mu(z, t)\alpha_z^{\pm}, \tag{17b}$$

where

$$\omega(z, t) = D^0 - A^0 - A^1 z + D^{-1}/z,$$

$$I^{+}(t) = \sum_{j=-\infty}^{+\infty} \{[D^{-1} + (2j+1)\mu^0] R_j S_{j-1} - [A^1 - 2(j+1)\mu^2] R_j S_j$$
$$- [D^{-1} + (2j+1)\mu^0] Q_j T_j + (A^1 - 2j\mu^2) Q_j T_{j-1}\},$$

$$I^{-}(t) = \sum_{j=-\infty}^{+\infty} \{(D^{-1} + 2j\mu^0) R_j S_{j-1} - [A^1 - (2j+1)\mu^2] R_j S_j$$
$$- [D^{-1} + 2(j+1)\mu^0] Q_j T_j + [A^1 - (2j+1)\mu^2] Q_j T_{j-1}\}.$$

The equation satisfied by $\beta^{\pm}(z, t)$ can be solved by the method of characteristics, starting from the initial data

$$\beta_0^{\pm}(z) = \beta^{\pm}(z, 0).$$

We thus obtain

$$\beta^{\pm}(z, t) = \beta_0^{\pm}[z_0(z, t)] \exp\left\{\pm \int_0^t \mathrm{d}t'\, \omega[\zeta(z_0(z, t), t'), t']\right\}, \tag{18}$$

the function $\zeta(z_0, t)$ being defined by the differential equation

$$\zeta_t(z_0, t) = -\mu[\zeta(z_0, t), t] \tag{19a}$$

with the boundary conditions

$$\zeta(z_0, 0) = z_0; \qquad \zeta(z_0, t) = z. \tag{19b}$$

To solve completely the IST we must know the evolution of the bound state parameters $(z_{(k)}^{(\pm)}, C_{(k)}^{(\pm)})$ which is obtained from equations (18, 19), taking into account equation (15); we obtain

$$(z_{(k)}^{(\pm)})_t = -\mu(z_{(k)}^{(\pm)}, t) \tag{20a}$$

$$C_{(k)}^{(\pm)}(t) = C_{(k)}^{(\pm)}(0)\zeta_{z_0}(z_0, t)|_{z_0=z_{(k)}^{(\pm)}(0)} \exp\left\{\pm\int_0^t \mathrm{d}t'\, \omega[z_{(k)}^{(\pm)}(t'), t']\right\}. \tag{20b}$$

We stress here that, due to the presence of $\mu(z, t)$, the eigenvalues evolve in time and thus the flow defined by equations (6) is no longer isospectral.

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